1206 Last Homework Sol^{n}.

#1. a) $I = \int x\sqrt{5+x^2}\,dx$

Set $x = \sqrt{5}\tan(\theta)$

$dx = \sqrt{5}\sec^2(\theta)\,d\theta$

So, $I = \int \sqrt{5}\tan(\theta)\sqrt{5+5\tan^2\theta}\,\sqrt{5}\sec^2(\theta)\,d\theta$

$= 5\sqrt{5}\int \tan(\theta)\sec^3(\theta)\,d\theta$

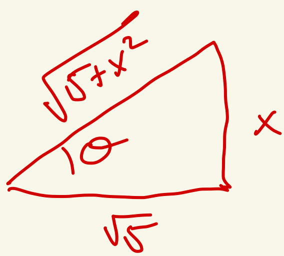
Set $u = \sec(\theta)$

$du = \sec(\theta)\tan(\theta)\,d\theta$

$I = 5\sqrt{5}\int u^2\,du$

$= \dfrac{5\sqrt{5}}{3}u^3 + C$

$= \dfrac{5\sqrt{5}}{3}\sec^3(\theta) + C$



$= \dfrac{5\sqrt{5}}{3}\left(\dfrac{\sqrt{5+x^2}}{\sqrt{5}}\right)^3 + C$

$= \dfrac{1}{3}(5+x^2)^{3/2} + C$

(b) $I = \displaystyle\int \frac{x}{(1-x^2)^{5/2}}\,dx$

Set $x = \sin(\theta)$

$dx = \cos(\theta)\,d\theta$

So $I = \displaystyle\int \frac{\sin(\theta)}{(1-\sin^2(\theta))^{5/2}} \cos(\theta)\,d\theta$

$= \displaystyle\int \frac{\sin(\theta)}{\cos^4(\theta)}\,d\theta$

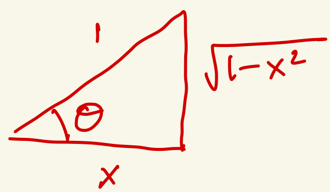
Set $u = \cos(\theta)$

$-du = \sin(\theta)\,d\theta$

So $I = -\displaystyle\int u^{-4}\,du$

$= \frac{1}{3} u^{-3} + C$

$= \frac{1}{3} \csc^3(\theta) + C$



$= \frac{1}{3} \cdot \frac{1}{\left(\sqrt{1-x^2}\right)^3} + C$

$= \frac{1}{3} (1-x^2)^{-3/2} + C.$

c) $\displaystyle\int \frac{x}{\sqrt{x^2-3}}\,dx$ Set $x = \sqrt{3}\sec(\theta)$

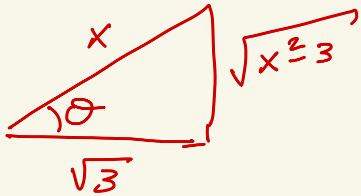
$dx = \sqrt{3}\sec(\theta)\tan(\theta)\,d\theta$

$$= \int \frac{\sqrt{3}\sec(\theta)}{\sqrt{3}\tan(\theta)} \cdot \sqrt{3}\sec(\theta)\tan(\theta)\,d\theta$$

$$= \sqrt{3}\int \sec^2(\theta)\,d\theta$$

$$= \sqrt{3}\tan(\theta) + C$$

$$= \sqrt{x^2-3} + C$$



#2. a) $\dfrac{1+x^4}{x^2+x}$ Need to divide!

$$\begin{array}{r|l} & x^2 - x + 1 \\ \hline x^2+x & 1 + x^4 \\ & x^3 + x^4 \\ \hline & 1 - x^3 \\ & -x^2 - x^3 \\ \hline & 1 + x^2 \\ & x + x^2 \\ \hline & 1 - x \end{array}$$

So, $\dfrac{1+x^4}{x^2+x} = x^2 - x + 1 + \dfrac{1-x}{x(x+1)}$

$$* \quad \frac{1-x}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{(A+B)x + A}{x(x+1)}$$

So
- $A+B = -1$
- $A = 1 \longrightarrow B = -2$

i.e. $\dfrac{1+x^4}{x^2+x} = x^2 - x + 1 + \dfrac{1}{x} - \dfrac{2}{x+1}$.

b) $$\frac{2x-4}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

$$= \frac{\left(A(x-1)+Bx\right)(x+2) + Cx(x-1)}{x(x-1)(x+2)}$$

So, $2x-4 = \left(A(x-1)+Bx\right)(x+2) + Cx(x-1)$

- $x=1$: $-2 = 3B \rightarrow B = -2/3$
- $x=0$: $-A(2) = -4 \rightarrow A = 2$.
- $x=-2$: $-8 = C(-2)(-3) = -6c \rightarrow C = \frac{-8}{6} = \frac{-4}{3}$.

$$\frac{2x-4}{x(x-1)(x+2)} = \frac{2}{x} - \frac{2/3}{x-1} - \frac{4/3}{x+2}$$

c) $$\frac{2x}{(x+1)(x^2+1)^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$= \frac{A}{x+1} + \frac{(Bx+C)(x^2+1)+Dx+E}{(x^2+1)^2}$$

$$= \frac{A(x^4+2x^2+1) + \left[Bx^3+Bx+Cx^2+C+Dx+E\right](x+1)}{(x+1)(x^2+1)^2}$$

$$= Ax^4+2Ax^2+A+Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex$$
$$+Bx^3+Bx+Cx^2+C+Dx+E \quad \Big/ (x+1)(x^2+1)^2$$

$$= (A+B)x^4 + (B+C)x^3 + (2A+B+C+D)x^2$$

$$+ (B+D+C+E)x + A+C+E \Big/ (x+1)(x^2+1)^2$$

So

$$\left.\begin{array}{l} \bullet\ A+B=0 \\ \bullet\ B+C=0 \end{array}\right\} \quad \begin{array}{l} A=-B \\ A=C \end{array}$$

$\bullet\ 2A+B+C+D=0 \rightarrow 2A+D=0 \rightarrow D=-2A$

$\bullet\ B+D+C+E=2 \rightarrow D+E=2 \rightarrow -2A+E=2$

$\bullet\ A+C+E=0 \longrightarrow 2A+E=0$

$\hookrightarrow 2E=2 \rightarrow E=1$

$\rightarrow A=-1/2$

$\rightarrow B=1/2$

$\rightarrow C=-1/2$

$\rightarrow D=1$

So,

$$\frac{2x}{(x+1)(x^2+1)^2} = \frac{-1/2}{x+1} + \frac{1/2\,x - 1/2}{x^2+1} + \frac{x+1}{(x^2+1)}$$

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