Sample!

Math 1206 Final Exam

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1. Show all of your work and determine if the following series converge or diverge.

$$\sum_{n=1}^{\infty} \frac{n^3}{n^3+1}, \quad \sum_{n=1}^{\infty} \frac{e^n}{n!}, \quad \sum_{n=0}^{\infty} \frac{3^n - 4^n}{9^n}$$

2. Consider the region R enclosed by the curves $y = \sqrt{x-1}$, $y = 1$, $x = 0$, $y = 0$.
   (a) Give a rough sketch the region R.
   (b) Find the area of R.

3. Find both the radius and interval of convergence for the series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{3ne^n}$$

4. Find a power series representation for $f(x) = \dfrac{1}{x}$ centred at $x = 3$. Be certain to mention where your formula is valid.

5. Find the Taylor series for $f(x) = \ln\left(\dfrac{x}{3}\right)$ centred at $x = 3$.

6. Integration!! Show all of your work.
   (a) $\displaystyle\int \sin^3(t)\cos^3(t)\ dt$
      i. Evaluate the integral using the substitution $u = \sin(t)$.
      ii. Evaluate the integral using the substitution $u = \cos(t)$.
      iii. Your answers in parts i. and ii. were different. Can you explain why?
   (b) Evaluate

$$\int \frac{t^3}{\sqrt{t^2+1}}$$

   and show all of your work.
   (c) Find a partial fraction decomposition for

$$R(x) = \frac{2x+1}{x^2(x-2)}.$$

   Using this decomposition, evaluate

$$\int R(x)\ dx.$$

7. How large should n be in order for a midpoint rule approximation of $\int_0^1 \ln(x^2+1)\ dx$ to be within 10^{-3}?

8. Arclength.
   (a) [5] Find the arclength of the curve $f(x) = \frac{2}{3}(x-1)^{3/2}$ on the interval $[1,2]$.

9. A particle travels along with acceleration given by $a(t) = t - t^2$. If the velocity at $t = 0$ is $v(0) = 1$ and displacement at time $t = 0$ is $p(0) = 3/2$. Find the function that gives the displacement of the particle at time t. (Recall that $p'(t) = v(t)$ and $v'(t) = a(t)$)

10. Determine if the series
$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3n}{3n^2-1}$$
converges absolutely, converges conditionally, or diverges. (Mention any tests you use to make your determinations)

11. Some Volumes.

    (a) Find the volume of the solid S obtained by rotating the region between the curves $y = x^3$ and $y = 2 - x^3$ over $[0, 1]$ about the horizontal line $y = 0$.

    (b) Find all values of p so that the volume of the infinitely long solid Q obtained by rotating the region between $y = \frac{1}{x^p}$ and the x-axis over $[1, \infty)$ about the x-axis is finite.

Solutions:

#1. • $\sum_{n=1}^{\infty} \frac{n^3}{n^3+1}$ Since $\lim_{n\to\infty} \frac{n^3}{n^3+1} = 1 \neq 0$, this series diverges by the test for divergence.

• $\sum_{n=1}^{\infty} \frac{e^n}{n!}$ $\lim_{n\to\infty} \frac{e^{n+1}}{(n+1)!} \Big/ \frac{e^n}{n!}$

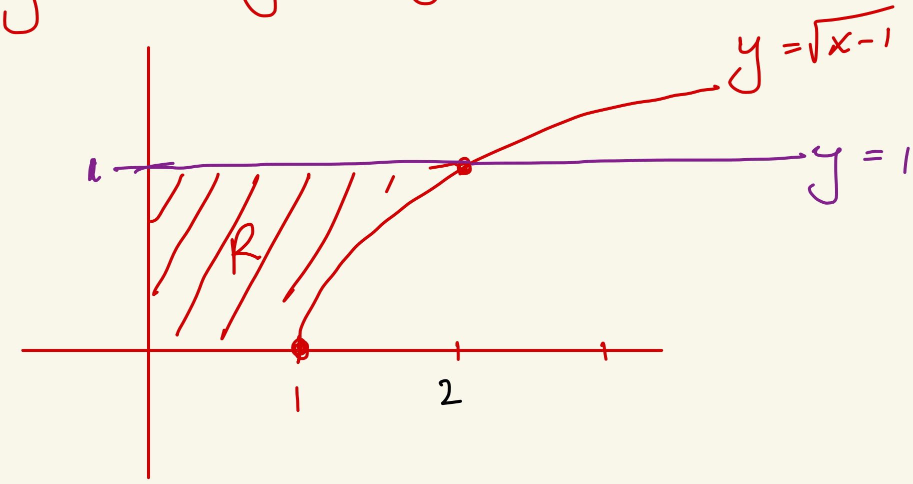
$= \lim_{n\to\infty} \frac{e}{n+1} = 0 < 1$. This series converges by the Ratio Test.

• $\sum_{n=0}^{\infty} \frac{3^n - 4^n}{9^n} = \sum_{n=0}^{\infty}\left(\frac{3}{9}\right)^n - \sum_{n=0}^{\infty}\left(\frac{4}{9}\right)^n$

$= \frac{1}{1-3/9} - \frac{1}{1-4/9}$ since $\frac{3}{9} < 1$, $\frac{4}{9} < 1$ series are convergent geom. series.

$= \frac{9}{6} - \frac{9}{5}$.

#2. $y=\sqrt{x-1},\ y=0,\ y=1,\ x=0$



$$A=\int_0^2 |f(x)-g(x)|\,dx=\int_0^1 dx+\int_1^2 (1-\sqrt{x-1})\,dx$$

$$=1+\left(x-\frac{2}{3}(x-1)^{3/2}\Big|_1^2\right)$$

$$=1+2-\frac{2}{3}(1)^{3/2}-1$$

$$=2-\frac{2}{3}$$

$$=\frac{4}{3}\text{ sq. units.}$$

#3. $\sum_{n=1}^{\infty}\frac{(x-3)^n}{3ne^n}$; $\lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right|=\lim_{n\to\infty}\frac{3ne^n}{3(n+1)e^{n+1}}$

$$=\frac{1}{e}$$

So, our R.O.C. is $R=e$ and the open interval of convergence is

$(3-e, 3+e)$

#4. Center $x_0 = 3$; $f(x) = 1/x$.

$$* \quad \frac{1}{x} = \frac{1}{3+(x-3)} = \frac{1}{3}\left(\frac{1}{1-\left(\frac{-(x-3)}{3}\right)}\right)$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{-(x-3)}{3}\right)^n \quad \text{if } \left|\frac{x-3}{3}\right| < 1 \text{ by Geom. Series}$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n(x-3)^n}{3^n} \quad \text{if } |x-3| < 3$$

i.e. if $x \in (0,6)$.

#5. Taylor Series centered at $x=3$; $f(x) = \ln\left(\frac{x}{3}\right)$

$f(x) = \ln\left(\frac{x}{3}\right)$; $f(3) = 0$

$f'(x) = \frac{1}{x}$; $f'(3) = 1/3$

$f''(x) = -\frac{1}{x^2}$; $f''(3) = -1/3^2$

$f'''(x) = \frac{2!}{x^3}$; $f'''(3) = \frac{2!}{3^3}$

$f^{IV}(x) = \frac{-3!}{x^4}$; $f^{IV}(3) = \frac{-3!}{3^4}$

$\vdots$

So,

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(3)}{n!}(x-3)^n$$

$$= 0 + \frac{1}{3}(x-3) - \frac{1!}{2!\,3^2}(x-3)^2 + \frac{2!}{3!\,3^3}(x-3)^3 - + \dots$$

$$= \sum_{n=1}^{\infty}\frac{(n-1)!}{n!\cdot 3^n}(x-3)^n\cdot(-1)^{n+1}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\cdot 3^n}(x-3)^n$$

#6. a) $\int \sin^3(x)\cos^3(x)\,dx$

(i) $u = \sin(x)$; $du = \cos(x)\,dx$

$$= \int u^3(1-u^2)\,du = \frac{u^4}{4} - \frac{u^6}{6} + C = \frac{\sin^4(x)}{4} - \frac{\sin^6(x)}{6} + C$$

(ii) $u = \cos(x)$; $du = \sin(x)\,dx$

$$= -\int u^3(1-u^2)\,du = \frac{u^6}{6} - \frac{u^4}{4} + C = \frac{\cos^6(x)}{6} - \frac{\cos^4(x)}{4} + C$$

(iii) Pythagorus: $\sin^4(x) = (1-\cos^2(x))^2 = 1 - 2\cos^2(x) + \cos^4(x)$

$\sin^6(x) = (1-\cos^2(x))^3 = 1 - 3\cos^2(x) + 3\cos^4(x) - \cos^6(x)$

giving $\frac{1}{4}\sin^4(x) - \frac{1}{6}\sin^6(x) = \frac{1}{4} - \frac{1}{2}\cancel{\cos^2(x)} + \frac{1}{4}\cos^4(x)$

$$-\frac{1}{6}+\frac{1}{2}\cancel{\cos^2(x)}-\frac{1}{2}\cos^4(x)+\frac{1}{6}\cos^6(x)$$

$$=\frac{2}{24}-\frac{1}{4}\cos^4(x)+\frac{1}{6}\cos^6(x)$$

$$=\frac{1}{6}\cos^6(x)-\frac{1}{4}\cos^4(x)+\frac{2}{24}$$

So, our 2 answers differ by only a constant!

#7. M_n : $\int_0^1 \ln(x^2+1)\,dx$; 10^{-3}.

$f(x)=\ln(x^2+1)$

So, $f'(x)=\dfrac{2x}{x^2+1}$ and we find $f''(x)=\dfrac{2(x^2+1)-4x^2}{(x^2+1)^2}$

$$=\frac{2-2x^2}{(x^2+1)^2}.$$

using the Δ-inequality

$$|f''(x)|\le\left|\frac{2}{(x^2+1)^2}\right|+\left|\frac{2x^2}{(x^2+1)^2}\right|$$

$\le 2+2x^2$ since $x^2+1\ge 1$ since $x\ge 0$.

$\le 2+2$ since $x\le 1$

$=4.$

So, by our formula

$$\Delta_{M_n}\le\frac{4(1-0)^3}{24n^2}=\frac{1}{6n^2}$$

$$\frac{1}{8n^2}<10^{-3}\longrightarrow n>\sqrt{\frac{1000}{6}}\simeq 12.9$$

therefore M_{13} approximates $\int_0^1 \ln(1+x^2)\,dx$ to 10^{-3}.

#8. arclength of $y=\frac{2}{3}(x-1)^{3/2}$ over $[1,2]$.

$y'=(x-1)^{1/2}\longrightarrow L=\int_1^2\sqrt{1+(x-1)}\,dx$

$$= \int_1^2 \sqrt{x}\,dx$$

$$= \frac{2}{3}x^{3/2}\Big|_1^2 = \frac{2\cdot 2^{3/2}}{3} - \frac{2}{3}$$

$$= \frac{2}{3}\left(2\sqrt{2}-1\right) \text{ units.}$$

#9. $a(t) = t - t^2$; $v(0) = 1$, $p(0) = 3/2$.

$$v(t) = \int a(t) = \frac{t^2}{2} - \frac{t^3}{3} + c$$

$$v(0) = 1 \longrightarrow c = 1$$

$$\longrightarrow v(t) = \frac{t^2}{2} - \frac{t^3}{3} + 1$$

$$p(t) = \int v(t)\,dt = \frac{t^3}{6} - \frac{t^4}{12} + t + D$$

$$p(0) = 3/2 \longrightarrow D = 3/2$$

We find the particle's position at time t:

$$p(t) = \frac{t^3}{6} - \frac{t^4}{12} + t + 3/2.$$

#10. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3n}{3n^2-1}$

- To check for absolute convergence, we analyze $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^n 3n}{3n^2-1}\right| = \sum_{n=1}^{\infty} \frac{3n}{3n^2-1}$

Now, $\displaystyle\frac{3n}{3n^2-1} \geq \frac{3n}{3n^2} = \frac{1}{n}$.

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is the divergent Harmonic Series, $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{3n^2-1}$ diverges by the comparison test and so our series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{3n^2-1}$ fails to converge absolutely.

• Since $\lim_{n\to\infty} \frac{3n}{3n^2-1} = 0$

and $\frac{d}{dx}\left(\frac{3x}{3x^2-1}\right) = \frac{3(3x^2-1) - 3x\cdot 6x}{(3x^2-1)^2}$

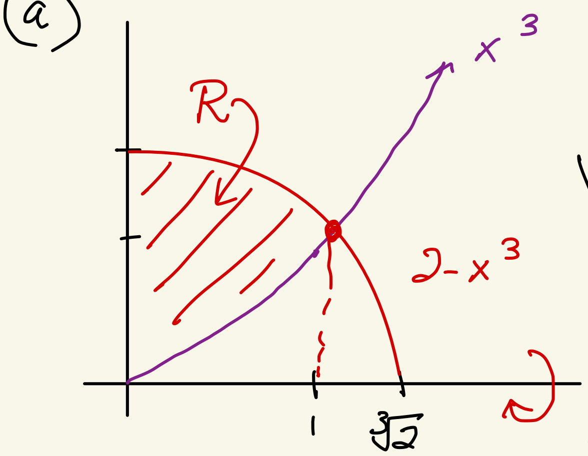
$= \frac{-1-9x^2}{(3x^2-1)^2} < 0$ for $x \geq 1$,

giving that $\left\{\frac{3n}{3n^2-1}\right\}_{n=1}^{\infty}$ is decreasing, the Alternating Series test shows that $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 3n}{3n^2-1}$ converges.

We conclude our series converges conditionally!

#11. x^3 and $2-x^3$ over $[0,1]$, ↻ about x-axis.

(a)



$V = \pi \int_0^1 \left((2-x^3)^2 - (x^3)^2\right) dx$

$= \pi \int_0^1 (4 - 4x^3 + x^6 - x^6)\, dx$

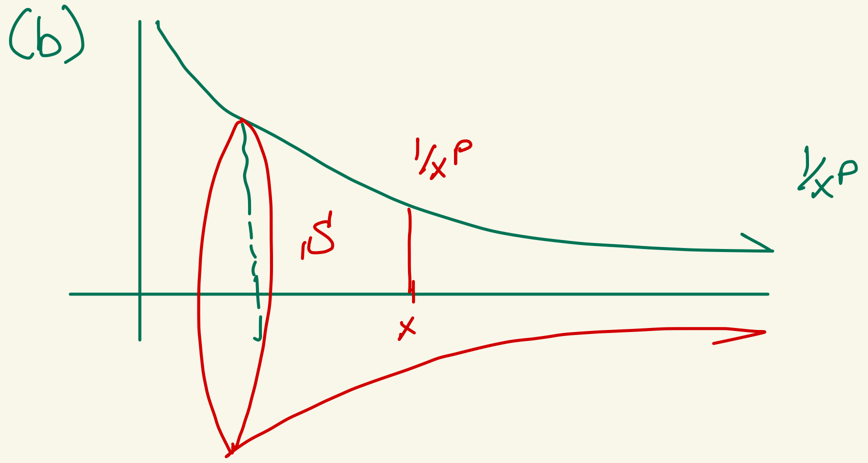
$= \pi \int_0^1 4 - 4x^3\, dx$

$= \pi \left(4x - x^4 \Big|_0^1\right)$

$= \pi(4-1)$

$= 3\pi$ cubic units.

(b)



$$V = \pi \int_1^\infty \frac{1}{x^{2p}}\,dx < \infty \text{ if } 2p > 1 \longrightarrow p > 1/2.$$

OR:

$$= \pi \lim_{c\to\infty} \begin{cases} \left.\frac{1}{-2p+1} x^{-2p+1}\right|_1^c & \text{if } p \neq 1/2. \\ \ln|x| \Big|_1^c & \text{if } p = 1/2 \end{cases}$$

$$= \pi \lim_{c\to\infty} \begin{cases} \frac{1}{-2p+1}\left(c^{-2p+1} - 1\right) & \text{if } p \neq 1/2 \\ \ln|c| & \text{if } p = 1/2 \end{cases}$$

$$= \pi \begin{cases} \frac{-1}{-2p+1} & \text{if } -2p+1 < 0 \\ \infty & \text{if } -2p+1 > 0 \\ \infty & \text{if } p = 1/2. \end{cases}$$

So, our volume is finite only if $-2p+1 < 0$ or $2p > 1 \longrightarrow p > 1/2$.

If $p > 1/2$ $V(S) = \frac{-\pi}{1-2p} = \frac{\pi}{2p-1}$ cubic units.