CAPE BRETON UNIVERSITY

MATH 1105 Final Exam Fall 2024

Solutions!

There may be errors!

1. This test is 20 pages long. There are 15 questions for a total of 93 points. It is your responsibility to check that you have a complete exam booklet. You have 180 minutes to complete this test.
2. All questions must be answered in the space provided. Indicate your responses clearly, showing all of your work.
3. If you require extra space for your answers, please use the extra pages before and after the question pages. **Clearly indicate** in the original question and in your continued work that you have continued your answer elsewhere.
4. **Do not write** on or mark the Crowdmark QR code at the top of each page. Doing so may cause exam pages to not be marked. Do **not** unstaple the exam booklet.
5. **No calculators or other electronic aides will be permitted.**

Please check that this has the correct name of your instructor.

Shannon Ezzat

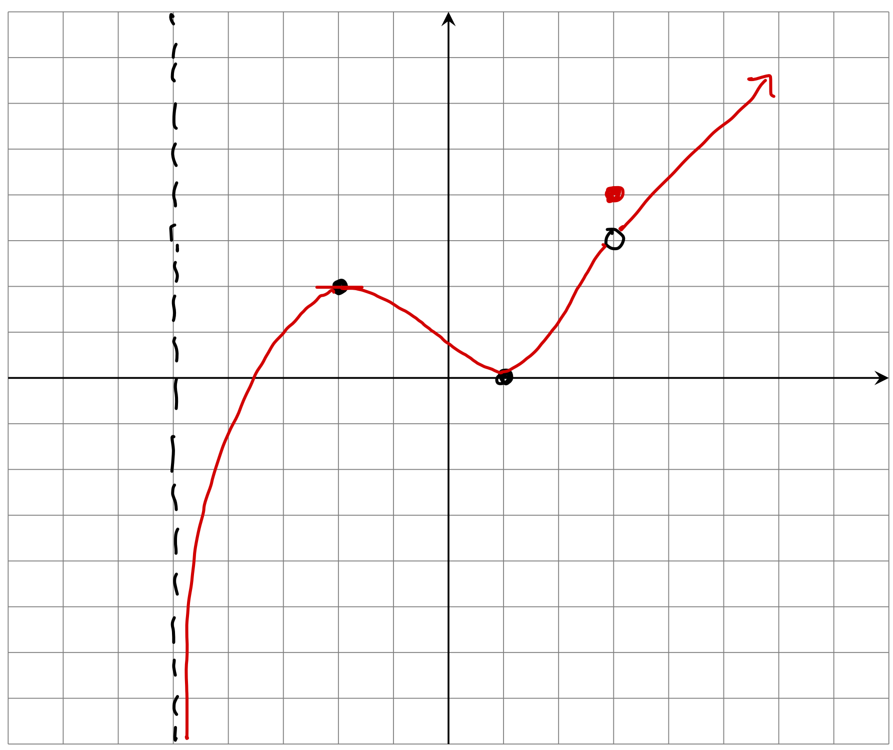
**You may use this space for scrap work or as extra space for your answer.**

**Please Note: If you are using this page as extra space, then please specify the question number below and indicate in the original question that you are continuing your answer here. Failure to do this may result in this page not being marked.**

Good Luck!

$\boxed{6}$ 1. (a) Draw a function $f(x)$ which has the following properties

1. $\lim_{x\to -5^+} f(x) = -\infty$.
2. $\lim_{x\to -2} f(x) = 2$
3. $\lim_{x\to -2} f'(x) = 0$ (Note the prime symbol here!)
4. $\lim_{x\to 1} f(x) = 0$
5. $\lim_{x\to 3} f(x) = 3$
6. $f(3) = 4$



$\boxed{2}$ (b) Give an example of a function $f(x)$ where $\lim_{x\to -\infty} f(x) = 0$ and $\lim_{x\to +\infty} f(x) = 0$. (Do not just draw a picture, give a formula)

$$f(x) = \frac{1}{1+x^2}$$

$\boxed{2}$ (c) With the same function you choose in b), if possible, determine $\lim_{x\to 0} f(x)$. If it is not possible, explain why not.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{1+x^2} = 1$$

$\boxed{5}$ 2. Using the definition of the derivative, compute the derivative function of $f(x) = \dfrac{1}{\sqrt{x}}$. You do not receive any points for calculating the derivative by any other method.

If $x = 0$,

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}$$

$$= \lim_{h\to 0} \frac{x - (x+h)}{h\sqrt{x}\sqrt{x+h}\,(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}\,(\sqrt{x} + \sqrt{x+h})}$$

$$= \frac{-1}{x \cdot 2\sqrt{x}}$$

$$= -\frac{1}{2x^{3/2}}$$

3. (a) Consider the function $f(x) = \sqrt{x} - 50\sin(x)$ on the interval $[1, 3]$. typo, use $[1,4]$

[1] i. Is this a continuous function on the given interval? Justify.

[2] ii. Does $f(x)$ have a root inside the interval $(1, 3)$? Justify.

[3] iii. If the answer to the last question was yes, find an interval of length $1/2$ that contains the root.

i) yes it's a polyⁿ added to a trig fcn!

ii) $f(1) = 1 - 50\sin(1) \simeq -41.07$ use radians!

and $f(4) = 2 - 50\sin(4) \simeq 39$

there is a root of $f$, say $c$, in $(1,4)$

iii) $f(3) \simeq -5.32$, our root is in $(3,4)$.

Since $f(3.5) \simeq 19.4$, the root is in $(3, 7/2)$

and this interval has length

$$L = \frac{7}{2} - 3 = 1/2.$$

$\boxed{5}$ 4. Find the equation of the tangent line of $f(x) = \arcsin(2x - 1)$ at $x = \frac{1}{2}$.

$f'(x) = \dfrac{1}{\sqrt{1-(2x-1)^2}} \cdot 2$ giving $f'(\frac{1}{2}) = \dfrac{2}{\sqrt{1-(1-1)^2}}$

$= 2.$

So, our tangent is of the form

$y = 2x + b.$

Since $f(\frac{1}{2}) = \sin^{-1}(0) = 0$, we have

$0 = 1 + b \to b = -1.$

Our tangent is $y = 2x - 1$.

5. Find the derivative of the following functions. You do not need to simplify your answer:

2 (a) $f(x) = \sqrt{x} - 4x^{\frac{5}{3}} - 3 + \frac{1}{2}x^{-2}$

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{20}{3}x^{2/3} - x^{-3}$$

2 (b) $f(x) = 2^x \arctan(x)$

$$f'(x) = \frac{2^x}{1+x^2} + 2^x \ln(2) \tan^{-1}(x).$$

2 (c) $h(t) = \cos(t)/(t^2 + 1)$

$$h'(t) = -\frac{(t^2+1)\sin(t) - 2t\cos(t)}{(t^2+1)^2}$$

2 (d) $k(s) = \ln(1 - s)$

$$k'(s) = -\frac{1}{1-s}$$

5 6. Let $f(x) = 2\pi x - x^2 + \sin(x)$. Show that $f(x)$ has a point $c$ where $f'(c) = 0$.

$f$ is cont. and diff$^{n}$ on $\mathbb{R}$ since it is a sum of poly's and trig fcns.

$$f(0) = 0 \text{ and } f(2\pi) = (2\pi)^2 - (2\pi)^2 + \sin(2\pi)$$
$$= 0.$$

So, Rolle's thm guarantees a $c \in (0, 2\pi)$ s.t.

$$f'(c) = 0.$$

$\boxed{5}$ 7. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ by implicit differentiation.

$$\sin(xy) = \cos(x + y)$$

differentiating implicitly:

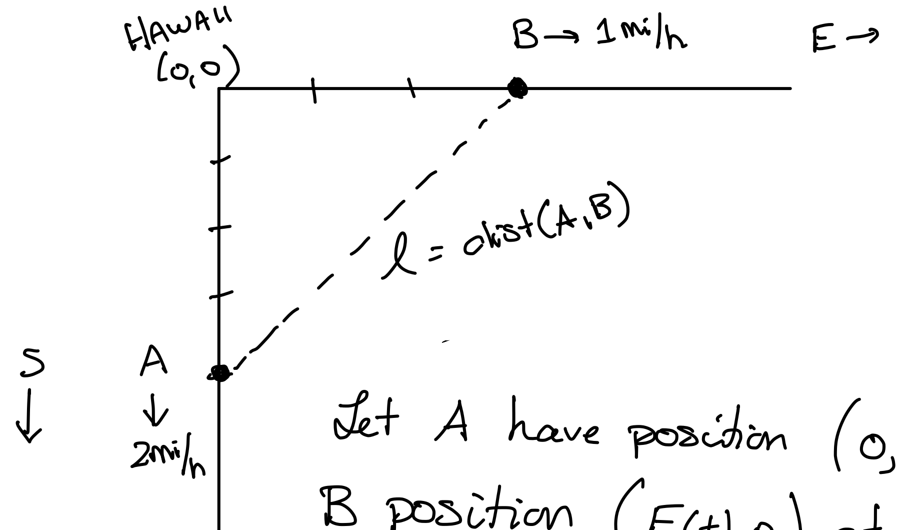
$$(y + xy')\cos(xy) = -(1+y')\sin(x+y)$$

So $xy'\cos(xy) + y'\sin(x+y) = -y\cos(xy) - \sin(x+y)$

$$y' = -\left(\frac{y\cos(xy) + \sin(x+y)}{x\cos(xy) + \sin(x+y)}\right)$$

yeesh...

$\boxed{5}$ 8. (a) Ship A is 4 miles directly south of Hawaii and is sailing south at a constant speed of 2 miles/hour. Ship B is 3 miles directly east of Hawaii and is sailing west at a constant speed of 1 mile/hour. At what rate is the distance between the ships changing? Assume Hawaii is a single point.




Let A have position $(0, S(t))$ and B position $(E(t), 0)$ at time $t$ with $S'(t) = -2$, $E'(t) = 1$. Then,

$$\ell(t) = \sqrt{S^2+E^2} \longrightarrow \ell'(t) = \frac{SS' + EE'}{\sqrt{S^2+E^2}} = \frac{-2S+E}{\sqrt{S^2+E^2}}$$

So, when $S=-4$, $E=3$, $\ell'(t) = \frac{8+3}{\sqrt{16+9}} = \frac{11}{5}$

$\boxed{2}$ (b) BONUS: If the situation in part (a) occurs at 12:00PM, at what time will the distance between the ships be increasing at a rate of 2 miles/hour?

We need $\ell'(t) = 2$

i.e. $-2S+E = 2\sqrt{S^2+E^2}$

i.e. $\cancel{4S^2} - 4SE + E^2 = \cancel{4S^2} + 4E^2$

i.e. $3E^2 + 4SE = 0$

i.e. $3E = -4S$

What are E and S? Position of Boat B: $(3+t, 0)$; A: $(0, -4-2t)$ $t$ hours after 12. So, $E = 3+t$, $S = -4-2t$

So $3E = -4S$ is

$9-16$

$9+3t = 16+8t \longrightarrow 5t = -7$

$\longrightarrow t = -7/5$ hrs

$16 - 9$

So, $\ell'(t) = 2$ 7/6 hrs earlier, at around 10:36.



9. (a) Find the following limits. If you wish to use l'Hôpital's rule, explain why you are allowed to use it in this situation.

$\boxed{1}$ i. $\lim_{x \to 1} \frac{x^3 - 2x^2 + 1}{x^3 - 1}$ $\left[\frac{0}{0}\right]$, So by l'Hôpital:

$$= \lim_{x \to 1} \frac{3x^2 - 4x}{3x^2 - 1} = \frac{3-4}{3-1} = -\frac{1}{2}.$$

$\boxed{1}$ ii. $\lim_{x \to \infty} \frac{x+1}{e^x}$ $\left[\frac{\infty}{\infty}\right]$, So by l'Hôpital

$$= \lim_{x \to \infty} \frac{1}{e^x} = 0$$

$\boxed{1}$ iii. $\lim_{x \to 0+} (1 + 3x)^{\frac{1}{x}}$ $[1^\infty]$

Setting $L = \lim_{x \to 0^+} (1+3x)^{1/x}$,

$$\ln(L) = \lim_{x \to 0^+} \frac{\ln(1+3x)}{x} \; ; \; \left[\frac{0}{0}\right]$$

by l'Hôpital: $\ln(L) = \lim_{x \to 0^+} \frac{3}{1+3x} = 3.$

So, $L = e^3$.

10. (a) Evaluate each of the following.

1 i. $\int \left(x^{2/3}+1\right)\ dx$

$$= \frac{1}{\frac{2}{3}+1} x^{2/3+1} + x + C$$

$$= \frac{3}{5} x^{5/3} + x + C$$

2 ii. $\int \left(\sin(x)+\sec^2(x)\right)\ dx$

$$= -\cos(x) + \tan(x) + C$$

3 iii. $\int xe^{x^2}\ dx$ Set $u = x^2$. Then $du = 2x\,dx$

and so $\frac{1}{2}\,du = x\,dx$

This gives $\int xe^{x^2}dx = \int e^{u}du = e^{u}+C = e^{x^2}+C.$

11. Let $f(x) = (x-1)e^x$, and let $f^{(n)}(x)$ be the $n$th derivative of $f(x)$.

$\boxed{2}$ (a) Identify a pattern in the $n$th derivatives of $f(x)$ to find $f^{(20)}(x)$.

$f'(x) = e^x + (x-1)e^x$

$f''(x) = e^x + e^x + (x-1)e^x = 2e^x + (x-1)e^x$

$f'''(x) = 2e^x + e^x + (x-1)e^x = 3e^x + (x-1)e^x$

$\vdots$

$f^{(n)}(x) = ne^x + (x-1)e^x$

So, $f^{(20)}(x) = 20e^x + (x-1)e^x = (x+19)e^x$.

$\boxed{1}$ (b) Is it always true that for any $f^{(n)}(x)$ there is a $z \in \mathbb{R}$ such that $f^{(n)}(z) = 0$? Justify your answer.

yes. If $f^n(x) = ne^x + (x-1)e^x = 0$

then $n + x - 1 = 0$ giving $x = 1-n$.

$\boxed{1}$ (c) Write down any function $g(x)$ where $g^{(n)}(x) = 0$ for all $n \geq 4$ but $g^{(3)}(x) \neq 0$; that is, $g^{(3)}(x)$ is not the zero function.

$g(x) = x^3$.

$g'(x) = 3x^2$

$g''(x) = 6x$

$g'''(x) = 6$

$g^{(iv)}(x) = 0$

$\downarrow$

$g^{(n)}(x) = 0$

So, $x^3$ is our candidate!

12. Strontium-90 has a half-life of 28 days.

$\boxed{2}$ (a) A sample has a mass of 50 mg initially. Give a formula for the mass remaining after $t$ days. You need not evaluate any expression that requires a calculator. (Hint: the general form of the equation is $m(t) = m(0)\mathrm{e}^{kt}$ where $k$ is a constant you must find).

$m(t) = 50e^{kt}$

Since $t_{1/2} = \frac{1}{k}\ln(1/2)$ we find

$k = \frac{1}{t_{1/2}}\ln(1/2) = \frac{1}{28}\ln(\tfrac{1}{2})$

So, $m(t) = 50e^{\frac{t}{28}\ln(1/2)}$

$\boxed{2}$ (b) How long does it take the sample to decay to a mass of 2mg? You need not evaluate any expression that requires a calculator.

$m(t) = 2$ if

$50e^{\frac{t}{28}\ln(1/2)} = 2$

i.e. $t = \dfrac{28}{\ln(1/2)}\ln(1/25)$

$\simeq 130$ days

or near 4.64 ½-lives.

$\boxed{8}$ 13. Determine the absolute maximum and minimum values of the function below on the given interval.

$$f(x) = \frac{x^2 + x + 1}{x^2 + 1}, \quad \text{for} \quad 0 \leq x \leq 2.$$

$$f'(x) = \frac{(2x+1)(x^2+1) - 2x(x^2+x+1)}{(x^2+1)^2}$$

$$= \frac{x^2+1-2x^2}{(x^2+1)^2}$$

$$= \frac{1-x^2}{(x^2+1)^2} = 0 \text{ if } x = \pm 1$$

we only need $x = 1$.

• $f(0) = 1$, $f(2) = \frac{7}{5}$, $f(1) = \frac{3}{2}$

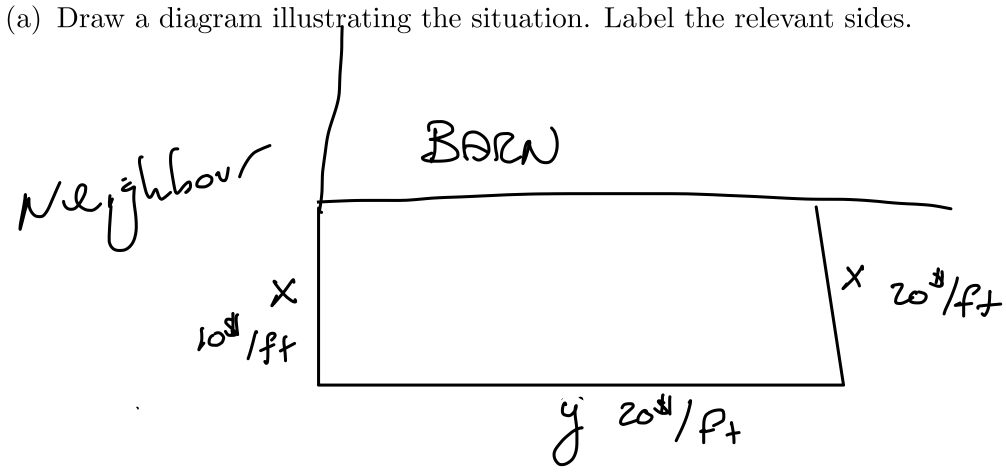
By the extreme value thm,

$f$ has abs max $3/2$ at $x = 1$

and abs. min $1$ at $x = 0$.

14. A farmer wants to fence in a rectangular plot of land adjacent to the north wall of his barn. *No fencing is needed along the barn.* The fencing costs \$20 per linear foot to install, and the fencing along the west side of the plot is shared with a neighbour who will split the cost of that portion of the fence.

2 (a) Draw a diagram illustrating the situation. Label the relevant sides.




2 (b) If the farmer has a budget of \$5000, find an equation that relates the farmer's budget to the side lengths of the fenced in area. Use the same variables as you chose in part (a).

Cost is

$$C(x,y) = 10x + 20x + 20y = 30x + 20y = 5000$$

i.e. $y = \dfrac{5000 - 30x}{20}$.

2 (c) Find the dimensions for the plot that would enclose the most area and use the farmer's budget.

Area: $A = x \cdot y = \dfrac{x}{20}(5000 - 30x)$ ; $x \in [0, \frac{5000}{30}]$

So $A' = \dfrac{1}{20}(5000 - 30x) - \dfrac{30}{20}x$

$= 250 - 3x = 0$ if $x = \dfrac{250}{3}$

and so since $A(0) = 0 = A\left(\frac{5000}{30}\right)$, the max area on budget is achieved with

$x = \dfrac{250}{3}$, $y = 125$

15. Let $f(x) = 1 - \frac{2}{x} + \frac{1}{x^2}$. Note that $f'(x) = \frac{2}{x^2} - \frac{2}{x^3}$ and $f''(x) = \frac{-4}{x^3} + \frac{6}{x^4}$

$\boxed{1}$ (a) Show that $f(x)$ can also be expressed as $f(x) = \left(\frac{x-1}{x}\right)^2$.

$$f(x) = \frac{x^2}{x^2} - \frac{2x}{x^2} + \frac{1}{x^2} = \frac{x^2 - 2x + 1}{x^2} = \frac{(x-1)^2}{x^2} = \left(\frac{x-1}{x}\right)^2$$

$\boxed{2}$ (b) Determine all vertical and horizontal asymptotes of the graph of $f(x)$.

Vertical: $x = 0$

Horizontal: $y = 1$ with $\lim_{x \to \pm\infty} f(x) = 1$

$\boxed{1}$ (c) Calculate all critical and singular points of $f$.

$$f'(x) = \frac{2}{x^2} - \frac{2}{x^3} = \frac{2x - 2}{x^3} = 0 \text{ if } x = 1$$

$\boxed{2}$ (d) Determine the intervals where $f(x)$ is increasing and where it is decreasing and the coordinates of any local minima or maxima.

Intervals: $(-\infty,0) \cup (0,1) \cup (1,\infty)$

$(-\infty,0)$ : $f'(-1) > 0$ ↗

$(0,1)$ : $f'(1/2) < 0$ ↘

$(1,\infty)$ : $f'(2) > 0$ ↗

$\boxed{1}$ (e) Calculate all points of inflection of $f$.

$$f''(x) = 2\left(\frac{1}{x^2} - \frac{1}{x^3}\right)'$$

$$= 2\left(-\frac{2}{x^3} + \frac{3}{x^4}\right)$$

$$= 2\left(\frac{-2x+3}{x^4}\right) = 0 \text{ if } x = 3/2,$$

$\boxed{2}$ (f) Determine the intervals where $f(x)$ is concave up and where it is concave down and the coordinates of any inflection points.

$(-\infty,0)$ : $f''(-1) > 0$ $\cup$

$(0,3/2)$ : $f''(1) > 0$ $\cup$

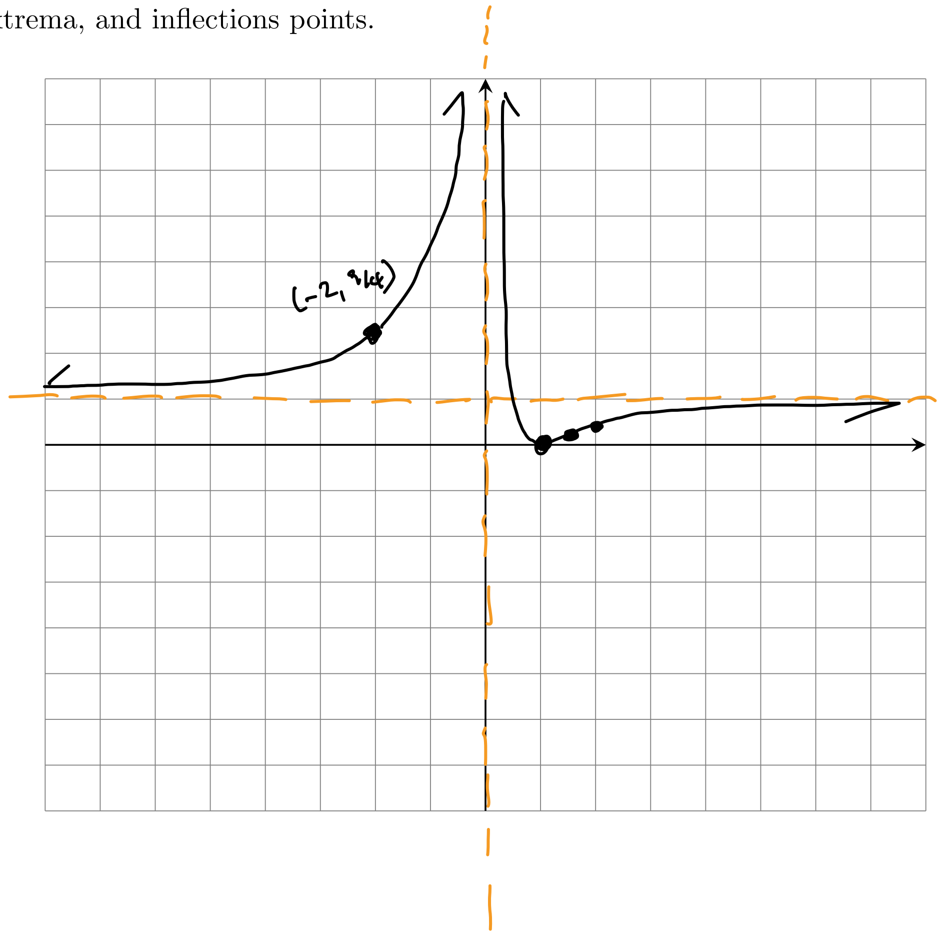
$(3/2,\infty)$ : $f''(2) < 0$ $\cap$

$f(1) = 0$, $f(-1) = 4$, $f(3/2) = \left(\frac{1/2}{3/2}\right)^2 = \frac{1}{9}$ $f(2) = \frac{1}{4}$

$f(-2) = 9/4$



$\boxed{2}$ (g) Sketch the graph of $f(x)$ in the coordinate system below, indicating any asymptotes, local extrema, and inflections points.



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