A Counterexample to Boundedness of Solutions of Poisson's Equation

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Abstract

In this paper, we explore Example 1.9 of [2] while filling in necessary details. This example shows that the Dirichlet Problem for Poisson's Equation on the unit ball $B = B(0,1)$ in $\mathbb{R}^n$,

$$\begin{cases} -\Delta u &= f \text{ in } B \\ u &= 0 \text{ on } \partial B \end{cases}$$

may admit arbitrarily large solutions. That is, given $0 < s < \frac{n}{2} - 1$, $C > 0$ and the Young function $A(t) = t^{\frac{n}{2}} \log(e - 1 + t)^s$, there is a continuous function $f \in L^A$ so that the corresponding solution u of the Dirichlet Problem on the unit ball satisfies $\|u\|_\infty > C\|f\|_{L^A}$. In doing so, we conclude that no Trudinger-type inequality of the form $\|u\|_\infty \leq C\|f\|_{L^A}$ can hold for a constant C independent of u and f when f is not sufficiently smooth.

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I would like to thank my supervisor and professor, Dr. Scott Rodney, whose enthusiastic support and impressive knowledge base provided much of the encouragement and resources necessary for the completion of this thesis.

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Contents

Notation

- n is the dimension of the space in which we are working. We always assume $n > 2$.
- $B = B(0, 1)$ is the unit ball in $\mathbb{R}^n$.
- Ω represents an open bounded connected set, also known as a domain. B is an example of such a set.
- ∂E indicates the boundary of a set E.
- The volume of any subset E of Ω is given by

$$|E| = \int_E dx.$$

- ω_n is the volume of B. That is, $\omega_n = |B| = \dfrac{2\pi^{\frac{n}{2}}}{n\Gamma\left(\frac{n}{2}\right)}$, where Γ is Euler's gamma function.
- For an integrable function f, the average value of $f(x)$ on E is denoted

$$⨍_E f(x)dx = \frac{1}{|E|} \int_E f(x)dx.$$

- $A(t)$ always denotes a Young function, which is an increasing function with certain special properties. Of particular interest to us are the facts that $A(0) = 0$ and A is convex on $[0, \infty)$. That is, for all $a, b \in [0, \infty)$, $t \in (0, 1)$,

$$A(ta + (1-t)b) \leq tA(a) + (1-t)A(b).$$

Or, if $A \in C^2([0, \infty))$, it must satisfy $A''(t) > 0$ for all $t \in [0, \infty)$. For further detail, we refer the reader to [1]. We will often focus on $A_s(t) = t^{\frac{n}{2}} \log(e - 1 + t)^s$ for $s > 0$.

- For $1 \leq p < \infty$, the L^p-norm (which we will be referring to as the p-norm) of a measurable $f : \Omega \to \mathbb{R}$ is given by $\|f\|_{L^p(\Omega)} = \left(\fint_\Omega |f(t)|^p dt \right)^{\frac{1}{p}}$. When there is no risk of confusion, we will write $\|f\|_{L^p(\Omega)} = \|f\|_p$.

- $L^p(\Omega)$ is the collection of all measurable functions whose p-norm on Ω is finite.

- Given $A(t)$, the Orlicz norm associated to A of a measurable function $f : \Omega \to \mathbb{R}$ is denoted $\|f\|_{L^A(\Omega)}$. When there is no risk of confusion, we will write $\|f\|_{L^A(\Omega)} = \|f\|_A$.

- $L^A(\Omega)$ is the set of all measurable $f : \Omega \to \mathbb{R}$ satisfying $\|f\|_{L^A(\Omega)} < \infty$.

- For any differentiable function $f : \Omega \to \mathbb{R}$, the gradient of f is $Df = (f_{x_1}, ..., f_{x_n})$.

- For any twice-differentiable f, $\Delta f = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2} = Div(Df)$.

- For $x > 0$, $\log(x)$ refers to the natural logarithm of x, also known as $\ln(x)$.

- $Lip_0(\Omega)$ is the collection of all Lipschitz functions with compact support in Ω.

Introduction

In their 2020 paper [2], David Cruz-Uribe and Scott Rodney studied *a priori* boundedness properties of weak solutions to the Drichlet Problem for Poisson's Equation

$$\begin{cases} -\Delta u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with $n > 2$.

Although their theorem is written generally for degenerate Laplace-like operators of the form $Xu = -Div(Q\nabla u)$ for a symmetric nonnegative definite matrix function Q defined in Ω, we present a simplified form of their result relevant to our purposes in this thesis.

In the literature associated to problem 2.1, it is the classical result of Serrin and Trudinger that one always has a bounded solution u to this problem whenever $f \in L^q(B)$ with $q > \frac{n}{2}$. More, there is a constant $C = C(q)$ independent of u, f so that the classical Trudinger inequality holds:

$$\|u\|_\infty \leq C\|f\|_q.$$

for any $q > \frac{n}{2}$ with a counterexample for the case $q = \frac{n}{2}$. However, in the theory of Orlicz spaces, there is a neat theorem. That is, if $f \in L^{\frac{n}{2}}(B)$, then there is a Young function $N(t) > t^{\frac{n}{2}}$ so that $f \in L^N(B)$. This indicates that it is possible to make the Trudinger-Serrin result better on the scale of Orlicz spaces. The theorem of Cruz-Uribe and Rodney is as follows.

Theorem 2.0.1. *Consider the log-bump Young function* $A(t) = t^{\sigma'} \log(e+t)^q$ *where* $q > \sigma' = \frac{n}{2}$ *is fixed. If* $f \in L^A(\Omega)$*, then any non-negative weak subsolution* $u \in H_0^1(\Omega)$ *of 2.1 satisfies*

$$\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^A(\Omega)}, \tag{2.2}$$

where C *is independent of both* $\boldsymbol{u}$ *and* f*.*

This result is very interesting because, as we will see in our explorations in this paper (primarily section 3.2), there are functions f such that $f \in L^{\frac{n}{2}}(B)$ and $f \in L^A(B)$, but $f \notin L^q(B)$ for any $q > \frac{n}{2}$.

The main tool exploited in their proof is the Sobolev inequality with "gain" $\sigma = \frac{n}{n-2}$. Specifically, there is a constant $C_0 \geq 1$ so that for every $\psi \in Lip_0(\Omega)$ one finds

$$\left(\int_\Omega |\psi(x)|^{2\sigma}\, dx\right)^{\frac{1}{2\sigma}} \leq C_0 \left(\int_\Omega |D\psi(x)|^2\, dx\right)^{\frac{1}{2}}. \tag{2.3}$$

This inequality is well known and a proof can be found in [4].

In this paper, we provide a detailed explanation of the counterexample to Theorem 2.0.1 given as Example 1.9 of [2]. Setting $\Omega = B$, we will demonstrate the following.

Theorem 2.0.2. *Let* $n > 2$ *and* $A(t) = t^{\frac{n}{2}} \log(e-1+t)^s$ *for a fixed* $s \in (0, \frac{n}{2}-1)$*. Then, given any constant* $c > 0$*, there exists a continuous function* $f \in L^A(B)$ *such that the solution* u *of*

$$\begin{cases} -\Delta u &= f \text{ for } x \in B, \\ u &= 0 \text{ for } x \in \partial B \end{cases}$$

satisfies $\|u\|_\infty > c\|f\|_A$*.*

This result shows that the result of Cruz-Uribe and Rodney cannot be extended to all $q > 0$ and establishes the range $\frac{n}{2} \geq q > \frac{n}{2} - 1$ as a focus of special interest to be explored in further research.

In order to achieve this goal, we will provide in Chapter 3 some background information and prior results concerning the tools used to develop the proof found in Chapter 4. In particular, section 3.1 contains the definition of L^p-norms, as well as a proof of Hölder's inequality, which we then use to show a comparison of relative sizes of p-norms (Theorem 3.1.2). This section primarily serves to provide a simpler case of some of our results for Orlicz norms. Then, section 3.2 motivates our use of Orlicz norms by presenting them as a generalization of p-norms, with Lemma 3.2.4 as a generalized version of Theorem 3.1.2. We will also explore some of the benefits of using Orlicz norms rather than p-norms. For example, we show with Lemma 3.2.4 and Example 3.2.3 that Orlicz norms provide a kind of "higher resolution" scale for measuring functions. To wrap up the preliminary information, we develop in section 3.3 a representation (3.12) of $u(0)$ with respect to f, where u and f satisfy the Drichlet Problem (2.1).

Finally, in chapter 4, we prove Theorem 2.0.2 by using Lemma 3.2.6 with a sequence of continuous functions by noting all entries in the sequence are bounded above by a function in an Orlicz class. This allows us to conclude the Orlicz norms of the sequence entries are uniformly bounded by the Orlicz norm of the upper bound function, which is finite. Additionally, we use (3.12) to show the sequence of solutions of the Drichlet Problem 3.3, which correspond to the functions in our first sequence, have Orlicz norms tending to ∞. Doing this completes the proof, and therefore, the thesis as a whole.

Preliminaries

This section demonstrates necessary background material for understanding the main result of this thesis. To ensure an effective presentation, the section begins by discussing L^p-norms and Hölder's inequality, then Young functions and Orlicz norms, and closing with Green's identities.

3.1 p-Norms

For $1 \leq p < \infty$, recall that $L^p(\Omega)$ is the collection of all $f : \Omega \to \mathbb{R}$ so that

$$\|f\|_{L^p(\Omega)} = \left(\fint_{\Omega} |f(t)|^p dt \right)^{\frac{1}{p}} < \infty.$$

More, if $p = \infty$, we let $L^\infty(\Omega)$ denote the set of all essentially bounded functions with norm given by the essential supremum. That is, given $f : \Omega \to \mathbb{R}$, $f \in L^\infty(\Omega)$ if

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in \Omega} |f(x)| < \infty.$$

Our first result in this section is Hölder's Inequality, which is proven in [7], but we provide a proof here as well, for the reader's benefit.

Theorem 3.1.1. *For* $1 \leq p \leq q \leq \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $f, g : \Omega \implies \mathbb{R}$,

$$\fint_{\Omega} f(x)g(x)dx \leq \|f\|_p \|g\|_q.$$

Proof. First, to limit possible values of $\|f\|_p$ and $\|g\|_q$, consider the case where $\|f\|_p = 0$. This means $f = 0$ a.e. and so $\|fg\|_1 = 0$, giving the desired inequality. Similar reasoning follows for the case where $\|g\|_q = 0$. So, for the remainder of the proof, we assume $\|f\|_p, \|g\|_q > 0$.

Now, let $\|f\|_p = \infty$. Then, $\|f\|_p\|g\|_q$ is infinite and the inequality holds. The same is true for $\|g\|_q = \infty$. Thus, we assume $0 < \|f\|_p, \|g\|_q < \infty$.

Now, we break this proof into two cases: (i) $p = 1$, $q = \infty$ and (ii) $1 < p \leq q < \infty$.

(i): Notice $|f(x)g(x)| \leq |f(x)|\|g\|_\infty$ for almost every x. Then, integrating both sides gives the desired inequality (since $\|g\|_\infty$ is a constant). That is,

$$\int_\Omega |f(x)g(x)|\ dx \leq \|g\|_\infty \int_\Omega |f(x)|\ dx = \|f\|_1\|g\|_\infty.$$

(ii): Now, let $F = \dfrac{f}{\|f\|_p}$, $G = \dfrac{g}{\|g\|_q}$, noticing that $\|F\|_p = \|G\|_q = 1$. By Young's inequality for products, given $a, b \geq 0$, $ab \leq \dfrac{a^p}{p} + \dfrac{b^q}{q}$. So, we have

$$|F(s)G(s)| \leq \frac{|F(s)|^p}{p} + \frac{|G(s)|^q}{q}.$$

Then, integrating over Ω gives

$$\begin{aligned}\int_\Omega |F(s)G(s)| ds &\leq \int_\Omega \frac{|F(s)|^p}{p} ds + \int_\Omega \frac{|G(s)|^q}{q} ds \\ &= \frac{\|F\|_p^p}{p} + \frac{\|G\|_q^q}{q} \\ &= \frac{1}{p} + \frac{1}{q} = 1\end{aligned}$$

Now, recalling the definitions of F and G, we find

$$\int_\Omega \frac{|f(s)g(s)|}{\|f\|_p\|g\|_q} ds \leq 1.$$

That is,

$$\fint_{\Omega} |f(s)g(s)|ds \leq \|f\|_p \|g\|_q.$$

□

One of the useful implications of Hölder's inequality is a simple relative size comparison between p-norms.

Theorem 3.1.2. *Let $1 \leq p < q \leq \infty$ and suppose $f \in L^q(\Omega)$. Then,*

$$\|f\|_{L^p(\Omega)} \leq \|f\|_{L^q(\Omega)}.$$

Proof. As in the proof of 3.1.1, we break this proof into two cases: (i) $q = \infty$ and (ii) $1 \leq p \leq q < \infty$.

(i): $\|f\|_p = \left(\fint_{\Omega} |f(t)|^p dt \right)^{\frac{1}{p}} \leq \left(\|f\|_{\infty}^p \fint_{\Omega} dt \right)^{\frac{1}{p}} = \|f\|_{\infty}.$

(ii): Let $\tau = \frac{q}{p}$ and set $\tau' = \dfrac{\frac{q}{p}}{\frac{q}{p} - 1} = \dfrac{q}{q-p}$, the conjugate exponent to τ. From there, we notice

$$\begin{aligned} \|f\|_p^p &= \fint_{\Omega} |f(t)|^p dt \\ &= \fint_{\Omega} |f(t)|^p \cdot 1 \ dt. \end{aligned}$$

Now, since $\dfrac{1}{\tau} + \dfrac{1}{\tau'} = 1$, Hölder's inequality gives

$$\begin{aligned} \fint_{\Omega} |f(t)|^p \cdot 1dt \leq \\ |f|^p\|_{\tau} \|1\|_{\tau'} \\ &= \left(\fint_{\Omega} |f(t)|^q dt \right)^{\frac{p}{q}} \left(\fint_{\Omega} 1 \ dt \right)^{\frac{q-p}{q}} \\ &= \|f\|_q^p. \end{aligned}$$

Taking the pth root of each side, we see $\|f\|_p \leq \|f\|_q$.

□

Noting that for f, p, q as in Theorem 3.1.2, we have $\|f\|_p \leq \|f\|_q < \infty$, we find

Corollary 3.1.3. *Let* $1 \leq p < q \leq \infty$ *and suppose* $f \in L^q(\Omega)$*. Then,*

$$f \in L^p(\Omega).$$

3.2 Orlicz Norms

In many cases, the scale of L^p-norms is not fine enough to precisely classify a function, and the work to follow is an example of such a case. To resolve this, we turn to the notion of Orlicz space as a generalization of L^p-space. Much of what we discuss has been illuminated by [1] and [6].

Like L^p-norms are defined with respect to a constant p, Orlicz norms are defined in relation to Young functions. Recall that given a Young function $A(t)$, $L^A(\Omega)$ is defined as the set of all $f : \Omega \to \mathbb{R}$ such that $\|f\|_{L^A(\Omega)} < \infty$, where the Luxembourg norm associated to $A(t)$ is given by

$$\|f\|_{L^A(\Omega)} = \inf \left\{ \lambda > 0 \ : \ \fint_\Omega A\left(\frac{|f(t)|}{\lambda}\right) \, dt \leq 1 \right\}. \tag{3.1}$$

For a given function f, we will refer to $\|f\|_{L^A(\Omega)}$ as the Orlicz norm of f on Ω.

To see how this generalizes L^p-spaces, notice that $L^p(\Omega) = L^A(\Omega)$ when $A(t) = t^p$. More, this gives $\|f\|_p = \|f\|_A$ for all $f \in L^p(\Omega)$. This result is apparent from inserting $A(t) = t^p$ into (3.1).

$$\begin{aligned}
\|f\|_{L^A(\Omega)} &= \inf\left\{\lambda > 0 \ : \ \fint_\Omega A\left(\frac{|f(t)|}{\lambda}\right)\ dt \le 1\right\} \\
&= \inf\left\{\lambda > 0 \ : \ \fint_\Omega \frac{|f(t)|^p}{\lambda^p}\ dt \le 1\right\} \\
&= \inf\left\{\lambda > 0 \ : \ \fint_\Omega |f(t)|^p\ \ dt \le \lambda^p\right\} \\
&= \inf\left\{\lambda > 0 \ : \ \left(\fint_\Omega |f(t)|^p\ \ dt\right)^{\frac{1}{p}} \le \lambda\right\} \\
&= \left(\fint_\Omega |f(t)|^p\ \ dt\right)^{\frac{1}{p}} = \|f\|_p.
\end{aligned}$$

For our purposes, we will only need to know whether the Orlicz norm of a given function is finite. Lemma 3.2.1 provides a useful criterion for determining this.

Lemma 3.2.1. *Let $f : \Omega \to \mathbb{R}$. Given a Young function $A(t)$, if there exists $\lambda \in (0, \infty)$ so that $A\left(\frac{|f(t)|}{\lambda}\right) \in L^1(\Omega)$, then $f \in L^A(\Omega)$.*

Proof. Let $A\left(\frac{|f(t)|}{\lambda}\right) \in L^1(\Omega)$. Then we have

$$\fint_\Omega A\left(\frac{|f(t)|}{\lambda}\right) dt = c < \infty.$$

If $0 < c \le 1$, we find $\|f\|_A \le \lambda$ by (3.1). But if $c > 1$, we note $\frac{1}{c} \in (0, 1)$ and since $A(0) = 0$, by the convexity of $A(t)$, we find

$$A\left(\frac{|f(t)|}{c\lambda}\right) \le \frac{1}{c} A\left(\frac{|f(t)|}{\lambda}\right).$$

So, we have

$$\fint_\Omega A\left(\frac{|f(t)|}{c\lambda}\right) dt \le \frac{1}{c} \fint_\Omega A\left(\frac{|f(t)|}{\lambda}\right) dt = 1.$$

Then, by (3.1), we conclude $\|f\|_A \le c\lambda < \infty$. ☐

Corollary 3.2.2. *Given $f : \Omega \to \mathbb{R}$ and a Young function $A(t)$, if*

$$\fint_{\Omega} A(|f(t)|)dt < \infty, \tag{3.2}$$

then $f \in L^A(\Omega)$.

Now we show further that Orlicz norms are a "finer scale" of norms. That is, for any two real numbers $1 \leq p < q$, there exist Orlicz norms associated to Young functions $A_s(t)$ such that $\|f\|_p \leq \|f\|_{A_s} \leq \|f\|_q$ for certain measurable functions f. We explore this explicitly in the following example, which follows the work in Example 19 of [5], but generalizing certain parameters.

Example 3.2.3. Let $r > 1, s > 0$, $1 < p < \infty$, $\Omega = \left(0, \frac{1}{e}\right)$, and set

$$f_r(t) = \frac{1}{t \log(e - 1 + \frac{1}{t})^r}$$

and

$$A_s(t) = t \log(e - 1 + t)^s.$$

Then,

(i) $f_r \in L^1(\Omega)$.

(ii) $f_r \notin L^p(\Omega)$ for any $p > 1$.

(iii) $f_r \in L^{A_s}(\Omega)$ exactly when $r - 1 > s > 0$.

To see why (i) holds, we make some pointwise estimates and then integrate using a change of variable. Indeed, for $t \in \Omega$, $\log\left(\frac{1}{t}\right) \geq 0$ and so,

$$\log\left(\frac{1}{t}\right)^r < \log\left(e - 1 + \frac{1}{t}\right)^r.$$

Multiplying by t, dividing and rearranging,

$$\frac{1}{t \log\left(e - 1 + \frac{1}{t}\right)^r} < \frac{1}{t \log\left(\frac{1}{t}\right)^r}.$$

Integrating both sides we find,

$$\int_0^{\frac{1}{e}} \frac{1}{t \log\left(e - 1 + \frac{1}{t}\right)^r}\, dt \leq \int_0^{\frac{1}{e}} \frac{1}{t \log\left(\frac{1}{t}\right)^r}\, dt.$$

Letting $u = \log\left(\frac{1}{t}\right)$, $du = -\frac{1}{t}\ dt$, we find

$$\int_0^{\frac{1}{e}} |f_r(t)|\ dt \leq -\int_\infty^1 \frac{1}{u^r}\ du = \int_1^\infty \frac{1}{u^r}\ du.$$

This last integral is finite only for $r > 1$.

These last calculations also help us to evaluate $\|f_r\|_p$, which allows us to show (ii). Using the same estimates and change of variable,

$$\int_0^{\frac{1}{e}} |f_r(t)|^p\ dt = \int_1^\infty \frac{e^{(p-1)u}}{u^{pr}}\ du.$$

Since $p > 1$, this integral is infinite as there is a constant $\beta = \beta(p) > 1$ so that $\dfrac{e^{(p-1)u}}{u^{pr}} \geq 1$ for $u \geq \beta$.

Lastly, to see (iii), we recall

$$A_s(t) = t\log(e - 1 + t)^s,$$

noting that A_s is a Young function for any $s > 0$. More,

$$A_s(|f_r(t)|) = \frac{1}{t\log(e-1+\frac{1}{t})^r}\log\left(e - 1 + \frac{1}{t\log(e-1+\frac{1}{t})^r}\right)^s.$$

Now, we do some estimation. For $t \in \Omega$, $\frac{1}{t} \geq 1$ and so $e-1+\frac{1}{t} \geq e$. Taking the log of both sides and taking the rth power, we get $\log(e-1+\frac{1}{t})^r \geq 1$. Multiplying both sides by t and rearranging,

$$\frac{1}{t\log(e-1+\frac{1}{t})^r} \leq \frac{1}{t}.$$

This then means

$$\log\left(e - 1 + \frac{1}{t\log(e-1+\frac{1}{t})^r}\right)^s \leq \log\left(e - 1 + \frac{1}{t}\right)^s.$$

Now, given this inequality, we have:

$$\begin{aligned}A_s(|f_r(t)|) &= \frac{1}{t\log(e-1+\frac{1}{t})^r}\log\left(e-1+\frac{1}{t\log(e-1+\frac{1}{t})^r}\right)^s\\ &\leq \frac{1}{t\log(e-1+\frac{1}{t})^r}\log\left(e-1+\frac{1}{t}\right)^s\\ &= \frac{1}{t\log(e-1+\frac{1}{t})^{r-s}}.\end{aligned}$$

Thus,

$$\int_0^{\frac{1}{e}} A_s(|f_r(t)|)dt \leq \int_0^{\frac{1}{e}} \frac{1}{t\log(e-1+\frac{1}{t})^{r-s}}dt.$$

And since $\int_0^{\frac{1}{e}} \frac{1}{t\log(e-1+\frac{1}{t})^{r-s}}dt < \infty$ when $r-s>1$, rearranging this inequality gives us $A(|f_r|) \in L^1(\Omega)$ when $0<s<r-1$. So, we conclude by Corollary 3.2.2 that $f_r \in L^{A_s}(\Omega)$ whenever $0<s<r-1$.

To help illustrate this example, we include the graphs of $f_2(t)$ and $A_{\frac{1}{2}}(|f_2(t)|)$ in Figures 3.1 and 3.2.

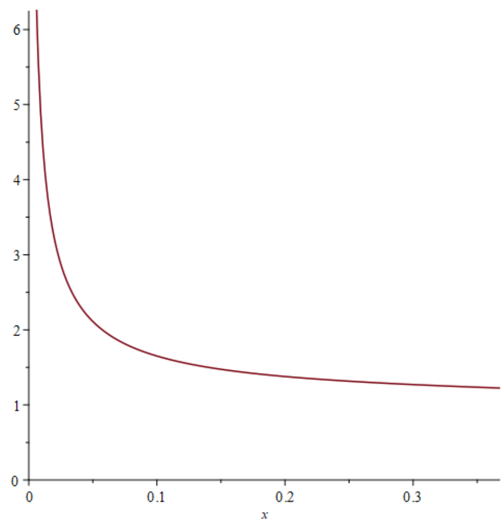

Figure 3.1: A graph of $f_2(t)$ on $(0, \frac{1}{e})$.

To understand some of the implications of this example, we now prove a lemma concerning the relative size comparison of Orlicz norms based on the relationship between the corresponding Young functions.

Lemma 3.2.4. *Let $A(t),\ B(t)$ be Young functions with $A(t) \leq B(t)$ for all $t \in [0, \infty]$. Then, for $f \in L^B(\Omega)$,*

$$\|f\|_A \leq \|f\|_B.$$

Proof. Since $A(t) \leq B(t)$ for $t \in [0, \infty]$, we have for any $\lambda > 0$ that

$$A\left(\frac{|f(t)|}{\lambda}\right) \leq B\left(\frac{|f(t)|}{\lambda}\right).$$

So, let

$$S = \left\{\lambda \ : \ \fint_\Omega A\left(\frac{|f(x)|}{\lambda}\right)\ dx \leq 1\right\}$$

and

$$T = \left\{\lambda \ : \ \fint_\Omega B\left(\frac{|f(x)|}{\lambda}\right)\ dx \leq 1\right\}.$$

Then, for $x \in \Omega,\ \lambda \in T$,

$$A\left(\frac{|f(x)|}{\lambda}\right) \leq B\left(\frac{|f(x)|}{\lambda}\right).$$

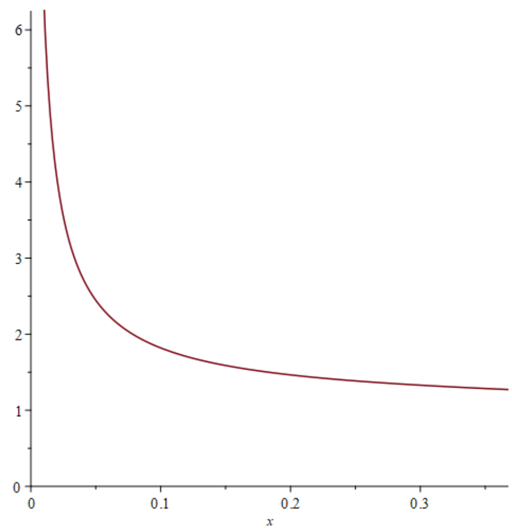

Figure 3.2: A graph of $A_{\frac{1}{2}}(|f(t)|)$ on $(0, \frac{1}{e})$.

Therefore, integrating over Ω, we find

$$\fint_{\Omega} A\left(\frac{|f(x)|}{\lambda}\right)\, dx \leq \fint_{\Omega} B\left(\frac{|f(x)|}{\lambda}\right)\, dx \leq 1$$

since $\lambda \in T$. So, we see $T \subset S$, thus $\inf(T) \geq \inf(S)$. Thus,

$$\|f\|_A \leq \|f\|_B.$$

□

Then, as with Theorem 3.1.2, we find

Corollary 3.2.5. *Let $A(t)$, $B(t)$ be Young functions with $A(t) \leq B(t)$ for all $t \in [0, \infty]$. Then,*

$$L^B(\Omega) \subset L^A(\Omega).$$

Now, noting $t \leq t\log(e-1+t)^s = A_s(t)$ for all $t \geq 0$ and recalling $L^1(\Omega) = L^A(\Omega)$ if $A(t) = t$, we see by the above corollary that $L^{A_s}(\Omega) \subset L^1(\Omega)$. Then, note that we found in Example 3.2.3 a function f_r satisfying $f_r \in L^{A_s}(\Omega)$ and $f_r \notin L^p(\Omega)$ for any $p > 1$. This helps to illustrate that there exist functions that are integrable to a degree that cannot be precisely conveyed on the scale of L^p-spaces. In fact, for a given $p > 1$, there exists

$s > 0$ such that $A_s(t) < t^p$ for t sufficiently large, so we may further conclude $L^p(\Omega) \subset L^{A_s}(\Omega) \subset L^1(\Omega)$. A proof of this claim may be found in [1].

To conclude this discussion of Orlicz norms, we prove a lemma which will be particularly useful in the final steps of the proof of our main theorem.

Lemma 3.2.6. *Let $g \in L^A(\Omega)$ for some Young function $A(t)$. Then, if f is a function so that $|f(x)| \leq |g(x)|$ for all $x \in \Omega$, then $\|f\|_A \leq \|g\|_A$.*

Proof. Since $A(t)$ is increasing on $[0, \infty]$, for any $\lambda > 0$ we have

$$A\left(\frac{|f(x)|}{\lambda}\right) \leq A\left(\frac{|g(x)|}{\lambda}\right)$$

for any $x \in \Omega$. This means that, given sets

$$G = \left\{\lambda \ : \ \fint_\Omega A\left(\frac{|g(x)|}{\lambda}\right)\ dx \leq 1\right\}$$

and

$$F = \left\{\lambda \ : \ \fint_\Omega A\left(\frac{|f(x)|}{\lambda}\right)\ dx \leq 1\right\},$$

we have for any $\lambda \in G$, that

$$\fint_\Omega A\left(\frac{|f(x)|}{\lambda}\right)\ dx \leq \fint_\Omega A\left(\frac{|g(x)|}{\lambda}\right)\ dx \leq 1.$$

So $\lambda \in F$, and we find $G \subset F$. Thus, $\inf(G) \geq \inf(F)$ and we see that the Luxembourg norms of f and g satisfy

$$\|f\|_{L^A(\Omega)} \leq \|g\|_{L^A(\Omega)}.$$

□

3.3 Green's Representation

As stated in Chapter 2, we are considering the Drichlet Problem for Poisson's Equation on the unit ball B in $\mathbb{R}^n$ for $n \geq 3$.

$$\begin{cases} -\Delta u &= f \text{ for } x \in B, \\ u &= 0 \text{ for } x \in \partial B \end{cases} \tag{3.3}$$

and looking for characteristics of f and u to find a pointwise representation of the solution u. Amazingly, this representation is given explicitly by f. To get there, we need two equations called Green's identities. To get to these, we undergo the following process, which is based on and borrows heavily from Section 2.4 of [3].

We begin with the divergence theorem on $B = B(0,1)$. Let $w = (w_1, ..., w_n)$ be any vector field in $C^1(\overline{B})$, and let ν be the unit outward normal vector to ∂B. The divergence theorem tells us that

$$\int_B \text{div } w \; dx = \int_{\partial B} w \cdot \nu \; ds. \tag{3.4}$$

Now let $u, v \in C^2(\overline{B})$, and note that the divergence theorem holds for B. Defining $w = vDu$ in (3.4) gives Green's first identity:

$$\int_B v\Delta u \; dx + \int_B Du \cdot Dv \; dx = \int_{\partial B} v\frac{\partial u}{\partial \nu} \; ds. \tag{3.5}$$

Swapping u and v in (3.5) gives

$$\int_B u\Delta v \; dx + \int_B Dv \cdot Du \; dx = \int_{\partial B} u\frac{\partial v}{\partial \nu} \; ds. \tag{3.6}$$

Then, subtracting (3.6) from (3.5) gives us Green's second identity:

$$\int_B (v\Delta u - u\Delta v) \; dx = \int_{\partial B} \left(v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right) ds. \tag{3.7}$$

Now, note Laplace's equation has the radially symmetric solution r^{2-n} for $n > 2$ and $\log(r)$ for $n = 2$, where r is the distance from a fixed point $y \in B$. So, let $r = |x - y|$. Then since $n > 2$, by [3], the normalized fundamental solution of Laplace's equation ($\Delta\Gamma = 0$) is

$$\Gamma(x - y) = \frac{1}{n(2-n)\omega_n}|x - y|^{2-n}. \tag{3.8}$$

This means

$$\Delta\Gamma(x - y) = \delta(y) = \begin{cases} 0 & \text{if } x \neq y \\ \infty & \text{if } x = y. \end{cases}$$

Now consider $B_\rho = B - \overline{B(y,\rho)}$, with $\rho > 0$ such that $B(y,\rho) \subset B$. Since $y \notin \overline{B_\rho}$, Γ has no singularities in $\overline{B_\rho}$, so we can substitute v for $\Gamma = \Gamma(x-y)$ in (3.7), giving us

$$\int_{B_\rho} \Gamma \Delta u \; dx = \int_{\partial B} \left(\Gamma \frac{\partial u}{\partial \nu} - u \frac{\partial \Gamma}{\partial \nu} \right) ds + \int_{\partial B(y,\rho)} \left(\Gamma \frac{\partial u}{\partial \nu} - u \frac{\partial \Gamma}{\partial \nu} \right) ds. \tag{3.9}$$

To make use of this, we look at the limits as $\rho \to 0$ of two terms in (3.9):

(i) $\displaystyle\int_{\partial B(y,\rho)} \Gamma \frac{\partial u}{\partial \nu} \; ds$: For $x \in \partial B(y,\rho)$, $|x-y| = \rho$ is constant, so we find

$$\left| \int_{\partial B(y,\rho)} \Gamma \frac{\partial u}{\partial \nu} ds \right| \leq |\Gamma(\rho)| \int_{\partial B(y,\rho)} \left| \frac{\partial u}{\partial \nu} \right| ds.$$

But noting $\nu \cdot Du$ is the directed derivative of u pointing out from $\partial B(y,\rho)$, we find

$$\begin{aligned} \left| \frac{\partial u}{\partial \nu} \right| &\leq |\nu \cdot Du| \\ &\leq |Du| \end{aligned}$$

since $|\nu| = 1$. Then, since u is a solution to (3.3) with f continuous, $u \in C^2(B)$. Thus, $|Du|$ is continuous on the closed sets $\overline{B(y,\tilde{\rho})}$ for any $0 < \tilde{\rho} \leq \rho$ and we find there exists $c > 0$ such that

$$|Du| \leq \sup_{x \in \overline{B(y,\rho)}} |Du| \leq c < \infty.$$

Then, letting $\tilde{c} = c|\Gamma(\rho)|$, we see

$$\begin{aligned} |\Gamma(\rho)| \int_{\partial B(y,\rho)} \left| \frac{\partial u}{\partial \nu} \right| ds &\leq \tilde{c} |\partial B(y,\rho)| \\ &= \tilde{c} \omega_n \rho^{n-1}. \end{aligned}$$

Finally, $\lim_{\rho \to 0} \tilde{c} \omega_n \rho^{n-1} = 0$, so we find

$$\lim_{\rho \to 0} \int_{\partial B(y,\rho)} \Gamma \frac{\partial u}{\partial \nu} ds = 0.$$

(ii) $\int_{\partial B(y,\rho)} u\frac{\partial \Gamma}{\partial \nu}\, ds$: As stated in (i), for $x \in \partial B(y,\rho)$, $\Gamma(x-y) = \Gamma(\rho)$. Thus,

$$\frac{\partial \Gamma}{\partial \nu} = \frac{\partial}{\partial \rho}\Gamma(\rho).$$

Then, by (3.8), we find

$$\frac{\partial}{\partial \rho}\Gamma(\rho) = \frac{1}{n\omega_n}\rho^{1-n}.$$

So,

$$\begin{aligned}\int_{\partial B(y,\rho)} u\frac{\partial \Gamma}{\partial \nu}\, ds &= \frac{1}{n\omega_n}\rho^{1-n}\int_{\partial B(y,\rho)} u\, ds\\ &= \fint_{\partial B(y,\rho)} u\, ds.\end{aligned}$$

Then, we have $\lim_{\rho\to 0}\fint_{B(y,\rho)} u\, ds = u(y)$ by Lebesgue's differentiation theorem, which can be found in [7].

Using the above work, we find that as $\rho \to 0$, rearranging (3.9) gives us Green's representation formula for $y \in B$,

$$u(y) = \int_{\partial B}\left(u\frac{\partial \Gamma}{\partial \nu}(x-y) - \Gamma(x-y)\frac{\partial u}{\partial \nu}\right)ds + \int_B \Gamma(x-y)\Delta u\, dx. \tag{3.10}$$

If u has compact support in B, then $u = 0 = \frac{\partial u}{\partial \nu}$ on ∂B. Using this with (3.10) gives

$$u(y) = \int_B \Gamma(x-y)\Delta u(x)\, dx. \tag{3.11}$$

For u satisfying (3.3), $-\Delta u = f$ on B, so (3.11) can be written

$$u(y) = -\int_B \Gamma(x-y)f(x)\, dx.$$

Letting $c_n = -\frac{1}{n(2-n)\omega_n}$ and $y = 0$ gives the following representation of $u(0)$:

$$u(0) = c_n\int_B \frac{f(x)}{|x|^{n-2}}dx. \tag{3.12}$$

We make use of this representation to prove our main theorem in the next section.

Proof of Theorem 2.0.2

Let $A_s(x)$ and s be as defined in 2.0.2. Then, let $A(x) = A_s(x)$ for ease of notation and set $f(x) = |x|^{-2}\log(e - 1 + |x|^{-1})^{-1}$ for $x \neq 0$ with $f(0) = 0$, noting that f is not continuous on B since $f(0) \neq \lim_{x\to 0} f(x)$. This forbids the use of (3.12) to represent $u(0)$ where u is the solution of the Drichlet Problem (3.3) corresponding to f. It also forbids the application of Theorem 2.0.2 to f and u. However, given any $k \in \mathbb{N}$, letting

$$\chi_k(x) = \begin{cases} 0, & 0 \leq x \leq 2^{-k-1} \\ 1, & 2^{-k} \leq x \leq 1. \end{cases} \tag{4.1}$$

with χ_k continuous, increasing and non-negative on $[2^{-k-1}, 2^{-k}]$ permits us to construct a sequence of continuous functions f_k defined as follows:

$$\begin{aligned} f_k(x) &= \chi_k(x)f(x) \\ &= \begin{cases} 0, & 0 \leq x \leq 2^{-k-1} \\ \chi_k(x)f(x), & 2^{-k-1} \leq x \leq 2^{-k} \\ f(x), & 2^{-k} \leq x \leq 1. \end{cases} \end{aligned}$$

Since f_k is continuous for each $k \in \mathbb{N}$, we can use (3.12) to write

$$u_k(0) = c_n \int_B \frac{f_k(x)}{|x|^{n-2}} dx \tag{4.2}$$

for each $k \in \mathbb{N}$. Here u_k is the solution to the Drichlet Problem (3.3) with $f = f_k$ (that is, u_k satisfies $-\Delta u_k = f_k$). Since f_k is a radial function, we can

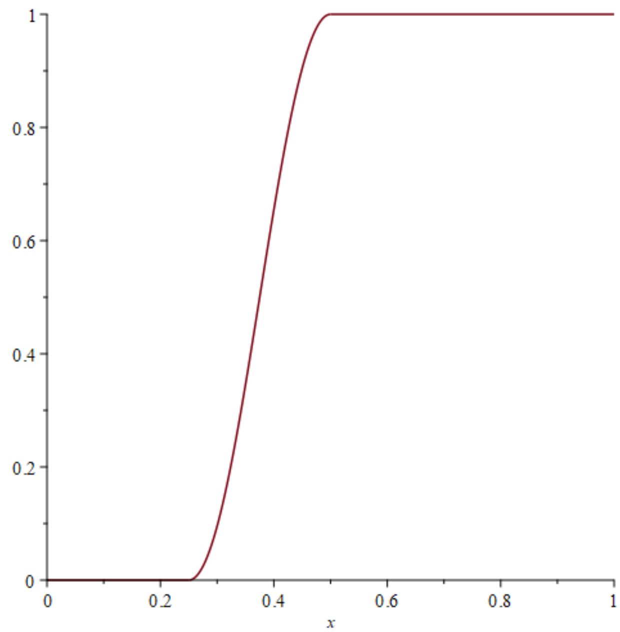

Figure 4.1: A possible graph of $\chi_1(x)$.

estimate our representation of $u_k(0)$ via a conversion to polar coordinates.

$$\begin{aligned} u_k(0) &= c_n \int_B \frac{f_k(x)}{|x|^{n-2}} dx \\ &= c_n \int_0^1 \frac{f_k(r)}{r^{n-2}} r^{n-1} dr \\ &\geq c \int_{2^{-k}}^1 \frac{r^{-2} \log(e-1+r^{-1})^{-1}}{r^{n-2}} r^{n-1} dr \\ &= c \int_{2^{-k}}^1 \frac{1}{r \log(e-1+\frac{1}{r})} dr. \end{aligned}$$

Now note as $k \to \infty$, $\displaystyle\int_{2^{-k}}^1 \frac{1}{r\log(e-1+\frac{1}{r})} dr \to \int_0^1 \frac{1}{r\log(e-1+\frac{1}{r})} dr.$

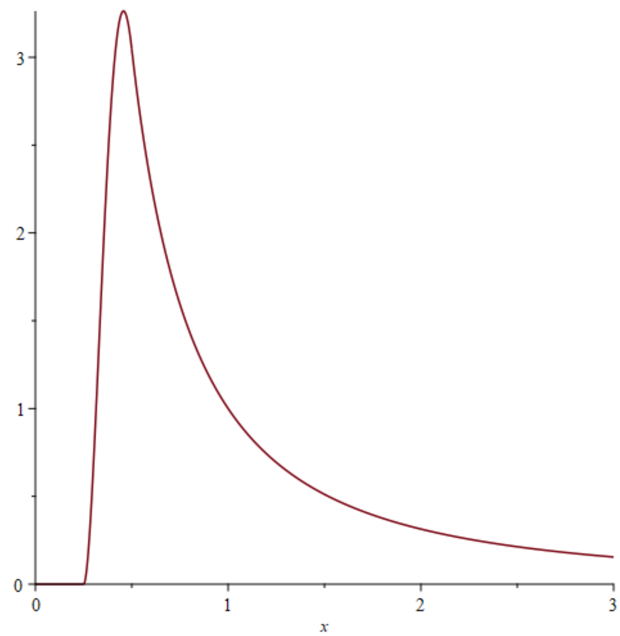

Figure 4.2: The graph of $f_1(x)$ given χ_1 as in Fig. 4.1.

Lemma 4.0.1. $\displaystyle\int_0^1 \frac{1}{r\log(e-1+\frac{1}{r})}dr = \infty.$

Proof. Noting that for any $r > 0$,

$$\begin{aligned}\log(e-1+r) &\leq \log(e-1+(e-1)r)\\ &= \log(e-1)+\log(1+r)\\ &\leq 1+\log(1+r).\end{aligned}$$

Multiplying both sides by r and rearranging, we find

$$\frac{1}{r\log(e-1+\frac{1}{r})} \geq \frac{1}{r\left(1+\log\left(1+\frac{1}{r}\right)\right)}.$$

Let $w = 1+\log(1+\frac{1}{r})$. Then, we can write $r = (e^{w-1}-1)^{-1}$ and

$$\begin{aligned}dw &= \frac{-1}{r^2(1+\frac{1}{r})}dr\\ &= \frac{-1}{r(r+1)}dr.\end{aligned}$$

Additionally,

$$\frac{d}{dw}\left(e^{w-1}-1\right) = (w-1)e^{w-1} > 0$$

for $w \in [1+\log(2), \infty) = I$, and

$$e^{(1+\log(2))-1} - 1 = 2 - 1 = 1,$$

so $(e^{w-1}-1)^{-1} \geq 0$ on I. Thus, $(e^{w-1}-1)^{-1} + 1 \geq 1$ on I. Using these estimates, we find

$$\begin{aligned}
\int_0^1 \frac{1}{r\log(e-1+\frac{1}{r})}dr &\geq \int_0^1 \frac{1}{r(1+\log(1+\frac{1}{r}))}dr \\
&= \int_0^1 \frac{r+1}{r(r+1)(1+\log(1+\frac{1}{r}))}dr \\
&= \int_{1+\log(2)}^{\infty} \frac{(e^{w-1}-1)^{-1}+1}{w}dw \\
&\geq \int_{1+\log(2)}^{\infty} \frac{1}{w}\, dw = \infty.
\end{aligned}$$

□

So, from our previous calculations, we see $\lim_{k\to\infty} \|u_k\|_\infty = \infty$.

To complete the proof, we now demonstrate the sequence $\{\|f_k\|_A\}$ is uniformly bounded.

To this end, we note that $f_k(x) \leq f(x)$ for all $k \in \mathbb{N}$, $t \in [0,1]$ since $\chi_k(x) \leq 1$. Thus, by Lemma 3.2.6, $\|f_k\|_A \leq \|f\|_A$ for all $k \in \mathbb{N}$, so it will suffice for us to show $\|f\|_A < \infty$, which we do now, using Corollary 3.2.2.

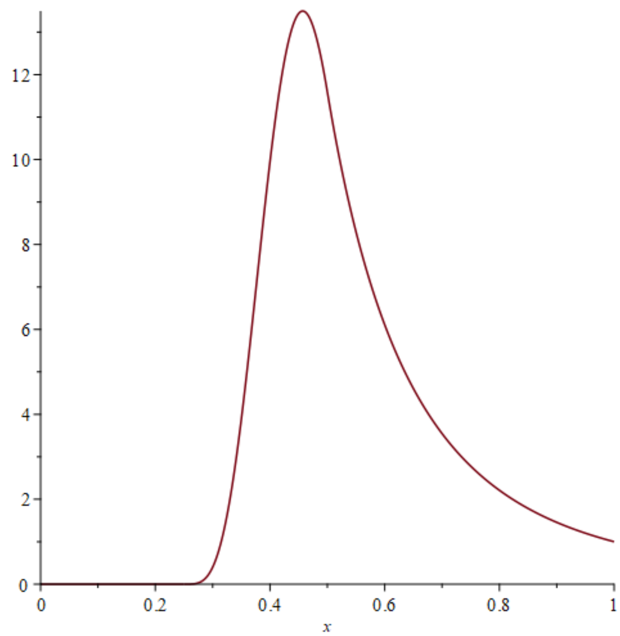

Figure 4.3: The graph of $A(|f_1(x)|)$ given f_1 as in Fig. 4.2.

Recall $A(x) = x\log(e-1+x)^s$ and $f(x) = |x|^{-2}\log(e-1+|x|^{-1})^{-1}$, so

$$\begin{aligned}
\int_B A(|f(x)|)dx &= \int_B \left| \left(\frac{1}{|x|^2 \log\left(e-1+\frac{1}{|x|}\right)} \right)^{\frac{n}{2}} \log\left(e-1+\frac{1}{|x|^2\log\left(e-1+\frac{1}{|x|}\right)} \right)^s \right| dx \\
&= \int_B \frac{1}{|x|^n \log\left(e-1+\frac{1}{|x|}\right)^{\frac{n}{2}}} \log\left(e-1+\frac{1}{|x|^2\log\left(e-1+\frac{1}{|x|}\right)} \right)^s dx \\
&= \int_0^1 \frac{1}{r^n \log\left(e-1+\frac{1}{r}\right)^{\frac{n}{2}}} \log\left(e-1+\frac{1}{r^2\log\left(e-1+\frac{1}{r}\right)} \right)^s r^{n-1}dr \\
&= \int_0^1 \frac{1}{r \log\left(e-1+\frac{1}{r}\right)^{\frac{n}{2}}} \log\left(e-1+\frac{1}{r^2\log\left(e-1+\frac{1}{r}\right)} \right)^s dr. \qquad (4.3)
\end{aligned}$$

To reach our intended estimate, we let $r \in (0,1)$, giving $e-1+\dfrac{1}{r} > e > 2$.

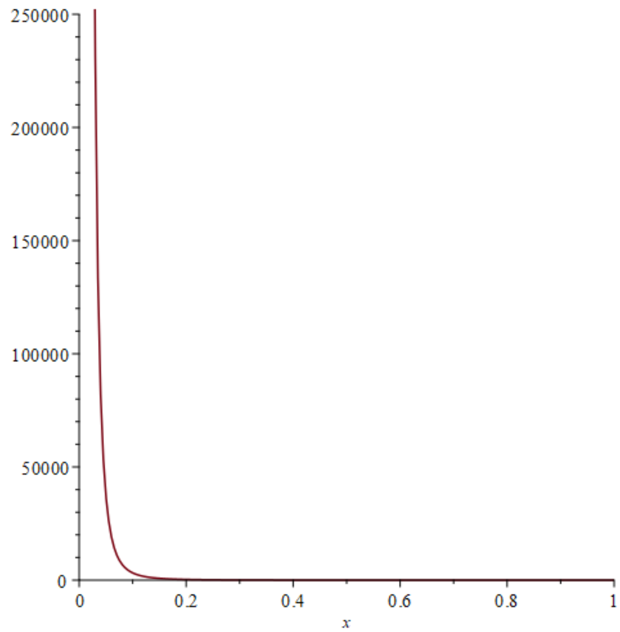

Figure 4.4: The graph of $A(|f(x)|)$.

Then,

$$\begin{aligned} 1 &< \log\left(e - 1 + \frac{1}{r}\right) \\ &= r^2\left(\frac{1}{r}\right)^2 \log\left(e - 1 + \frac{1}{r}\right) \\ &< r^2\left(e - 1 + \frac{1}{r}\right)^2 \log\left(e - 1 + \frac{1}{r}\right). \end{aligned}$$

Dividing, we find

$$\frac{1}{r^2 \log\left(e - 1 + \frac{1}{r}\right)} < \left(e - 1 + \frac{1}{r}\right)^2,$$

and so

$$e - 1 + \frac{1}{r^2 \log\left(e - 1 + \frac{1}{r}\right)} < 2\left(e - 1 + \frac{1}{r}\right)^2$$

since $e-1<\left(e-1+\frac{1}{r}\right)^2$. Then, since $2<e-1+\dfrac{1}{r}$, we have

$$e-1+\frac{1}{r^2\log\left(e-1+\dfrac{1}{r}\right)}<\left(e-1+\frac{1}{r}\right)^3.$$

Taking the log of both sides, we find

$$\log\left(e-1+\frac{1}{r^2\log\left(e-1+\dfrac{1}{r}\right)}\right)<3\log\left(e-1+\frac{1}{r}\right).$$

Inserting this estimate into (4.3) gives us

$$\begin{aligned}\int_B A(|f(x)|)dx <&3^s\int_0^1\frac{1}{r\log\left(e-1+\frac{1}{r}\right)^{\frac{n}{2}}}\log\left(e-1+\frac{1}{r}\right)^s dr\\ =&3^s\int_0^1\frac{1}{r\log\left(e-1+\frac{1}{r}\right)^{\frac{n}{2}-s}}dr.\end{aligned}$$

To conclude our proof, we note that

$$\int_{\frac{1}{e}}^1 A(|f(t)|)dt<\infty$$

since $A(|f(t)|)$ is continuous on $[\frac{1}{e},1]$.

Also, by our work in Example 3.2.3 (and letting $r=\frac{n}{2}$), we know

$$\int_0^{\frac{1}{e}} A(|f(t)|)dt<\infty,$$

since $0<s<\frac{n}{2}$ by assumption. Therefore,

$$\int_0^1 A(|f(t)|)dt<\infty.$$

So, by Corollary 3.2.2, we have $\|f\|_A<\infty$. □

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