4205 Homework #3 - Diffusion in a Rod and Fourier Series

Scott Rodney

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1. In this problem, you will solve the Neumann-Problem for the heat equation that limits flux at the edges of a rod. Indeed, consider

$$\begin{cases} u_t = ku_{xx}, & \text{for } x \in [0, L],\ t > 0 \\ u_x(0,t) = 0 = u_x(L,t), & \text{for } t \geq 0 \end{cases}$$

   (a) Using separation of variables, reinterpret the condition at the ends of the rod and rewrite your equation as a system of two differential equations - the Temporal and Spatial equations.

   (b) Find all eigenvalues and eigenfunctions for the spatial equation.

   (c) Now solve the temporal equation with the eigenvalues you found and write out the family of solutions you have $\{u_n(x,t)\}_{n\in\mathbb{N}}$ you found.

2. Show by direct calculation the following orthogonality relations when $m, n \in \{1, 2, 3, \dots\}$:

   - $\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0.$
   - $\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = 0 = \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx$ provided $n \neq m$.
   - $\int_{-\pi}^{\pi} \sin^2(nx)\,dx = \pi = \int_{-\pi}^{\pi} \cos^2(nx)\,dx$

3. Using your work in part 2 along with

$$\int_{-\pi}^{\pi} \sin(nx)\,dx = 0 = \int_{-\pi}^{\pi} \cos(nx)\,dx$$

for all $n = 0, 1, 2, \ldots$ you will prove a neat result. Suppose that

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} [A_n \cos(nx) + B_n \sin(nx)]$$

for every $x \in [-\pi, \pi]$. Show that

$$||f|| = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\ dx\right)^{1/2} = \sqrt{\frac{A_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left[A_n^2 + B_n^2\right]}$$