4205 Homework #5

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1. Consider the Heat problem on a thin square plate $P = [0,1] \times [0,2]$ with zero flux at the boundaries:

$$\begin{cases} u_t = -2\left(u_{xx} + u_{yy}\right) & ; \quad t > 0,\ (x,y) \in P \\ u_x(0,y,t) = 0 & ; \quad y \in (0,2),\ t > 0 \\ u_x(1,y,t) = 0 & ; \quad y \in (0,2),\ t > 0 \\ u_y(x,0,t) = 0 & ; \quad x \in (0,1),\ t > 0 \\ u_y(x,2,t) = 0 & ; \quad x \in (0,1),\ t > 0 \\ u(x,y,0) = \varphi(x,y) & ; \quad (x,y) \in P \end{cases}$$

   (a) Draw a picture of the domain and describe what this model problem describes.

   (b) Using separation of variables, find the spatial and temporal equations related to this problem along with their boundary conditions.

   (c) Using your result in (a), find all eigenvalues and eigenfunctions. Hint:You will find that $\lambda = 0$ and $\lambda_n = \pi^2\left(n^2 + \frac{m^2}{4}\right)$ are your eigenvalues. Remember to show that $\lambda < 0$ is not an eigenvalue.

   (d) using your results above, solve the problem and animate your solution in Maple for $\varphi(x,y) = xy(x-1)(y-2)$.

2. Some Sturm-Liouville Theory. Consider the S-L problem on $[1,b]$:

$$(x^2 Z')' + \lambda Z = 0$$

   with boundary conditions

$$Z(1) = 0 = Z(b).$$

   (a) Using the substitutions $Z = \dfrac{S}{\sqrt{x}}$ and $\lambda = 1/4 - \mu$, show that our problem is changed into

$$(xS')' + \frac{\mu}{x} S = 0$$

   with

$$S(1) = 0 = S(b).$$

   Hint: The equation is Cauchy-Euler!

(b) Using our results from class, can you find the Eigenvalues and Eigenfunctions associated to our original problem and their orthogonality relation?

(c) BONUS Using what you found, find an eigenfunction expansion for $f(x) = 1 - (x-2)^2$ on $(1, 3)$.